1. Income Taxes in an Optimal Growth Model

Consider a Ramsey model of an economy in a competitive equilibrium with technological growth. There is a representative household and a representative firm. The household's utility function is

$$U \equiv \int_0^\infty u(c_t)e^{-\rho t}dt,$$

with

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta},$$

where there is no population growth, and $\rho > 0$. The representative firm has a production function $F[K_t, A_t, L_t] = K_t^\alpha (A_t, L_t)^{1-\alpha}$. For simplicity, assume capital does not depreciate after production ($\delta = 0$). Further,

$$\dot{A}_t = gA_t$$

where A_t is the level of technology that grows at a constant rate g.

At every point in time, the government institutes an income tax. That is, the household pay a share τ of its total income to the government. Find the competitive equilibrium of this economy, using the following steps.

a. Write down representative household's maximization problem, solve it, and derive the 4 equations that characterize the solution. Explain in words, intuitively, what the Hamiltonian function means, and what the 4 equations represent.

b. Write down firm's maximization problem and the first-order conditions for this problem. Translate these conditions into intensive form. Derive the 2 equations that characterize the solution.

c. What are the equilibrium conditions for this economy? Derive the government budget constraint.

d. Combine your answers to parts a) – c) and derive a pair of differential equations for the variables c and k. Can you draw a phase diagram?

e. Define $\hat{k} = \frac{k}{A}$, $\hat{y} = \frac{y}{A}$, and $\hat{c} = \frac{c}{A}$. Derive a pair of differential equations for the variables $\hat{c}$ and $\hat{k}$. Draw the phase diagram, carefully identifying (and deriving mathematically) all the important points.

f. What is the growth rate of the economy? What are the growth rates of consumption and capital? Discuss. Draw the phase diagram, carefully identifying (and deriving mathematically) all the important points.

g. Do the following comparative dynamics exercise: $\tau' > \tau = 0$. That is, compare the economy with and without the income tax. As usual, the baseline economy starts in its steady state at time $t = 0$. The modified economy starts at time $t = 0$. Draw (i) the phase diagram for both cases, indicating what is different, and (ii) the time paths of c and k for both cases. Carefully discuss your results.

2. Some consumers evaluate consumption relative to the prices they observe, believing that higher prices signal higher quality or status. Consider a representative consumer with preferences represented by the utility function:

$$u(x,p) = \sum_{l=1}^{L} \log(x_l) + \gamma \sum_{l=1}^{L} \log(p_l), \quad \text{where } \boldsymbol{x} \in R^L_{++}, \boldsymbol{p} \in R^L_{++}, \gamma \in R.$$

The consumer takes prices p as given, and faces the standard budget constraint:

$$\boldsymbol{p} \cdot \boldsymbol{x} \leq w, \quad w > 0.$$

a. Give examples of goods that you think this type of utility function applies to and explain why.

b. Derive the Marshallian demand function $\boldsymbol{x}(\boldsymbol{p}, w)$ and show whether the Marshallian demand function is homogeneous of degree zero in prices and wealth.

c. Derive the indirect utility function $v(\boldsymbol{p}, w)$, and show whether the indirect utility function homogeneous of degree zero in $(\boldsymbol{p}, w)$.

d. Interpret the role of the parameter γ. For what values of γ is the utility function increasing in prices? What does this imply about the consumer's welfare under inflation (i.e., proportional increase in all prices and wealth)?

e. Formulate the dual (expenditure minimization) problem for a target utility level $\bar{u}$, and solve for the Hicksian demand function $\boldsymbol{h}(\boldsymbol{p}, \bar{u})$.

f. Derive the expenditure function $e(\boldsymbol{p}, \bar{u})$ and determine whether it is homogeneous of degree one in prices.

3. Two firms produce a homogeneous good with a market inverse demand for the good of $p(Q) = a - bQ$, where $Q = q_1 + q_2$, and q_i is the output for firm i. Firm 1 has high cost of production: $C_1(q_1) = c_H q_1$, and firm 2 has low costs: $C_2(q_2) = c_L q_2$, where $c_H > c_L$.

   a. Suppose firm 1 is the only firm in the market (firm 1 is a monopolist). Solve for the profit maximizing quantity for firm 1 and the profits. Call this q^M and π^M.

   b. Assume that both firms are in the market and making their decisions simultaneously.
      i. Suppose each firm can choose their own price (Bertrand). Solve for the Nash equilibrium price strategies and the resulting quantity and profits for each firm. Call these q_i^B and π_i^B.
      ii. Suppose each firm can choose their own quantities (Cournot). Solve for the Nash equilibrium quantity strategies and resulting profits for each firm. Call these q_i^C and π_i^C.

   c. Suppose firm 2 is uncertain about the inverse demand function. In particular, they do not know what the slope term, b, is precisely. They know that $E[b] = \mu$ and $E[b^2] - E[b]^2 = \sigma^2$. Firm 1 knows the value of b. Both firms will still make their quantity decisions simultaneously. Solve for the Bayesian Nash equilibrium strategies.

   d. Finally, suppose the firms will play an infinitely repeated simultaneous quantity choice game. The firms have decided that to cooperate, each firm will produce one-half of the monopoly quantity found in part (a) ($q_i = \frac{q^M}{2}$). [Calculate the profits for firm 2 from playing this quantity.]. Because firm 2 has lower cost, firm 1 will make a payment to firm 2 (call this X) in each period as an incentive to cooperate. [If firm 2 deviates from the cooperative strategy, then they do not get X (i.e., if $q_2 \neq \frac{q^M}{2}$)]. If either firm deviates from the cooperative strategy in any period, then they both assume that the Cournot quantities from part (b ii) will be played in all subsequent periods. If the discount factor is $\delta = \frac{3}{4}$, calculate the minimum and maximum values of X that can support cooperation as part of a subgame perfect equilibrium. Use the following parameter values: $a = 25$, $b = 2$, $c_L = 8$, $c_H = 9$. [Potentially helpful equations: $\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$, $\sum_{t=1}^{\delta} \delta^t = \frac{\delta}{1-\delta}$.]

4. Consider a 2×2 exchange economy. Consumer 1's utility function is $U_1(x_1, y_1) = 4x_1 - \frac{16}{y_1}$ and her endowment is $\omega_1 = (8, 0)$. Consumer 2's utility function is $U_2(x_2, y_2) = \ln(x_2) + \frac{y_2}{4}$ and his endowment is $\omega_2 = (12, 6)$.

   a. Derive the contract curve (the set of Pareto optimal allocations) for this economy. Draw an Edgeworth box showing the initial allocation, and the contract curve.

   b. Set up the optimization problem for consumer 1 and derive her offer curve (demand functions for x_1, y_1). Draw the offer curve for consumer 1 in your Edgeworth box.

   c. Set up the optimization problem for consumer 2 and derive his offer curve (demand functions for x_2, y_2). Draw the offer curve for consumer 2 in your Edgeworth box.

   d. Find the competitive equilibrium for this economy. This will be a price vector $p^\star$ together with an allocation $(x^\star, y^\star)$. Confirm that this outcome is Pareto optimal.